D-BRANES, B-FIELDS AND TWISTED K-THEORY

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Abstract. In this note we propose that D-brane charges, in the presence of a topologically non-trivial B-field, are classified by the K-theory of an infinite dimensional C^*-algebra. In the case of B-fields whose curvature is pure torsion our description is shown to coincide with that of Witten.

1. Introduction and discussion

Recently it was realized that, as a consequence of the fact that D-branes naturally come equipped with (Chan-Paton) vector bundles, D-brane charges take values in the K-theory of the spacetime manifold X, rather than in the integral cohomology $H^\bullet(X,\mathbb{Z})$, as one naively might have expected [MM, Wi]. In fact, it has been argued that the RR-fields themselves are classified by the K-theory of X as well [MW].

The identification of D-brane charges and K-theory classes has, among other things, led to a better understanding of the spectrum of D-branes, in particular of the existence of stable, nonsupersymmetric (i.e., non-BPS) D-branes [Se]. In many cases these novel stable, nonsupersymmetric D-branes can be understood as bound states of a brane-antibrane system with tachyon condensation. In particular, brane-antibrane annihilation in the case of $D9$-branes [Se] has been an important tool (and motivation) in Witten's work [Wi].

The identification of D-brane charges and K-theory discussed above holds in the case of a vanishing NS B-field. In the presence of a B-field the arguments of [Wi] need to be modified as is apparent, for instance, from the analysis of global string worldsheet anomalies [Wi, FW, Ka]. In addition, it is well-known that gauge fields on the D-brane in the presence of a (constant) B-field are more naturally interpreted as connections over noncommutative algebras rather than as connections on vector bundles (see, e.g., [CDS, SW] and references therein). Therefore it is natural to suspect that D-brane charges in the presence of a B-field should be identified with the K-theory of some noncommutative algebra. Recall that B-fields are topologically classified by the cohomology class of their field strength H, i.e., $[H] \in H^3(X,\mathbb{Z})$ (see, e.g., Section 6 in [FW] for a mathematical treatment of B-fields). In the case that $[H] \in Tors(H^3(X,\mathbb{Z}))$, i.e. $[H]$ represents a torsion class in $H^3(X,\mathbb{Z})$ (the case of a flat B-field), Witten has argued that the D-brane charges take values in a certain

twisted version of K-theory (see [Wi], Section 5.3) or, equivalently, in the K-theory of a certain noncommutative algebra over X, known as an Azumaya algebra. This proposal has been worked out and analyzed in more detail in [Ka]. What happens in the case when $[H] \neq 0$ is not torsion, i.e. in the case of D-branes in the presence of NS-charged backgrounds, has remained obscure so far.

The purpose of this note is to propose a candidate for the relevant K-theory in the more general case. We will argue that D-brane charges, in the presence of a topologically non-trivial B-field, are classified by the twisted K-theory of certain infinite-dimensional, locally trivial, algebra bundles of compact operators as introduced by Dixmier and Douady [DD]. The necessity of going to infinite dimensional algebra bundles, in order to incorporate nontorsion classes can, in a sense, be interpreted as going off-shell (as anticipated in Section 5.3 of [Wi]). The relevant twisted K-theory, which was defined by Rosenberg [Ros], is however not much 'bigger' than its finite-dimensional counterpart. In fact, we will show that even though the underlying C^*-algebra is infinite dimensional, its twisted K-theory has a finite dimensional (local coordinate) description. Furthermore, we will show that for $[H] \in Tors(H^3(X,\mathbb{Z}))$ our proposal is equivalent to that of [Wi, Ka]. The potential relevance of Dixmier-Douady theory to the classification of D-brane charges in the presence of topologically non-trivial B-fields was already noticed in [Ka]. However, in that paper this possibility was dismissed for the wrong reasons.

In the remainder of this section we briefly summarize our proposal. The relevant mathematical constructions and background material will be explained in more detail in the next few sections. A more detailed exposition of our proposal and its relevance for string theory will appear elsewhere [BM].

The starting point of the Sen-Witten construction of stable nonsupersymmetric D-branes (in type IIB String Theory), as remarked above, is a configuration of n $D9$ brane-antibrane pairs. When $[H] = 0$ the D-branes carry a principal $U(n)$ bundle, while for $[H] \neq 0$ the D-branes carry a principal $SU(n)/\mathbb{Z}_n = U(n)/U(1)$ bundle over X. Cancelation of global string worldsheet anomalies, however, requires $n[H] = 0$ [Wi, Ka, FW], i.e., requires $[H]$ to be a torsion element. To incorporate nontorsion $[H]$ we somehow need to take the limit $n \to \infty$. This leads us to the study of principal $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$ bundles over X where $\mathcal{H}$ is an infinite dimensional, separable, Hilbert space. [In fact, $\lim_{n\to\infty} SU(n)/\mathbb{Z}_n = PU(\mathcal{H})$ in a certain sense which will be made more precise in the paper.] It turns out that isomorphism classes of principal $PU(\mathcal{H})$ bundles over X are parametrized precisely by $H^3(X,\mathbb{Z})$. If we denote by $\mathcal{K}$ the C^*-algebra of compact operators on $\mathcal{H}$ one can identify $PU(\mathcal{H}) = Aut(\mathcal{K})$. It follows that isomorphism classes of locally trivial bundles over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$ are also parametrized by $H^3(X,\mathbb{Z})$. The cohomology class that is associated to a locally trivial bundle $\mathcal{E}$ over

X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$ is called the Dixmier-Douady invariant of $\mathcal{E}$ and is denoted by $\delta(\mathcal{E})$.

Our proposal is that D-brane charges, in the presence of nontrivial $[H] \in H^3(X, \mathbb{Z})$, are classified by the K-theory of the C^*-algebra $C_0(X, \mathcal{E}_{[H]})$ of continuous sections that vanish at infinity, of the algebra bundle $\mathcal{E}_{[H]}$ over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$ where $\delta(\mathcal{E}_{[H]}) = [H]$, i.e., by

$$K^j(X, [H]) \;=\; K_j(C_0(X, \mathcal{E}_{[H]}))\,, \qquad j = 0, 1\,. \tag{1}$$

This is precisely the twisted K-theory as defined by Rosenberg [Ros].

In the case when $[H]$ is a torsion element in $H^3(X, \mathbb{Z})$ the algebra $C_0(X, \mathcal{E}_{[H]})$, defined above, is Morita equivalent to the Azumaya algebra defined in [Wi, Ka] and therefore the K-theories of these algebras are canonically isomorphic.

Note that our proposal answers some of the questions raised at the end of Section 5.3 in [Wi] and refutes some of the remarks made in Section 7 of [Ka], concerning the case when the NS 3-form field H is a not a torsion element in $H^3(X, \mathbb{Z})$. We also mention that twisted K-theory has appeared earlier in the mathematical physics literature, in the study of the quantum Hall effect [CHMM, CHM].

2. Brauer groups and the Dixmier-Douady invariant

Let X be a locally compact, Hausdorff space with a countable basis of open sets, for example a smooth manifold. Then recall that the classifying space of the third cohomology group of X is the Eilenberg-Maclane space $K(\mathbb{Z}, 3)$, where $K(\mathbb{Z}, k)$ is defined uniquely upto homotopy as being the topological space with the property that $\pi_k(K(\mathbb{Z}, k), *) = \mathbb{Z}$ and $\pi_j(K(\mathbb{Z}, k), *) = 0$ for $j \neq k$ (cf. [Wh]). We will now describe a candidate for the Eilenberg-Maclane space $K(\mathbb{Z}, 3)$.

Let $\mathcal{H}$ denote an infinite dimensional, separable, Hilbert space and $\mathcal{K}$ the C^*-algebra of compact operators on $\mathcal{H}$. Let $U(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$. Then it is a fundamental theorem of Kuiper that $U(\mathcal{H})$ is contractible in the strong operator topology. Now define the projective unitary group on $\mathcal{H}$ as $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$, where $U(1)$ consists of scalar multiples of the identity operator on $\mathcal{H}$ of norm equal to 1. It follows that a model for the classifying space of $U(1)$ is $BU(1) = PU(\mathcal{H})$. Since $\mathbb{R}$ is contractible, it follows that $U(1) = \mathbb{R}/\mathbb{Z}$ is itself a $K(\mathbb{Z}, 1)$. Therefore $PU(\mathcal{H})$ is a model for $K(\mathbb{Z}, 2)$ and finally a model for $K(\mathbb{Z}, 3)$ is the classifying space of $PU(\mathcal{H})$, i.e., $K(\mathbb{Z}, 3) = BPU(\mathcal{H})$. We conclude that

$$H^3(X, \mathbb{Z}) = [X, K(\mathbb{Z}, 3)] = [X, BPU(\mathcal{H})], \tag{2}$$

where $[X, Y]$ denotes the homotopy classes of maps from X to Y. In other words, isomorphism classes of principal $PU(\mathcal{H})$ bundles over X are parametrized by $H^3(X, \mathbb{Z})$.

Now for $g \in U(\mathcal{H})$, let $Ad(g)$ denote the automorphism of $\mathcal{K}$ given by $T \to gTg^{-1}$. It is well known that Ad is a continuous homomorphism of $U(\mathcal{H})$ onto $Aut(\mathcal{K})$ with kernel $U(1)$, where $Aut(\mathcal{K})$ is given the point-norm topology (cf. [RW], Chapter 1). It then follows that $PU(\mathcal{H}) = Aut(\mathcal{K})$. Therefore the isomorphism classes of locally trivial bundles over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$ are also parametrized by $H^3(X, \mathbb{Z})$. Since $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$, we see that the isomorphism classes of locally trivial bundles over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$ form a group under the tensor product, where the inverse of such a bundle is the conjugate bundle. This group is known as the *infinite Brauer group* and is denoted by $Br^\infty(X)$ (cf. [Pa]). The cohomology class in $H^3(X, \mathbb{Z})$ that is associated to a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$ is called the *Dixmier-Douady invariant* of $\mathcal{E}$ and is denoted by $\delta(\mathcal{E})$, see [DD].

We next give a "local coordinate" description of elements in $Br^\infty(X)$. Let $\{\mathcal{U}_i\}$ be a good open cover of X, i.e. such that all $\mathcal{U}_i$ and their multiple overlaps are contractible. An element of the C^*-algebra $C_0(X, \mathcal{E})$, of continuous sections of $\mathcal{E}$ that vanish at infinity, is a collection of functions $R_i : \mathcal{U}_i \to \mathcal{K}$ such that on the overlaps $\mathcal{U}_i \cap \mathcal{U}_j$ one has

$$R_i = g_{ij} R_j g_{ij}^{-1} = Ad(g_{ij}) R_j. \tag{3}$$

Here $g_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \to U(\mathcal{H})$ are continuous functions on the overlaps, satisfying $g_{ij} g_{ji} = 1$, and therefore $Ad(g_{ij}) : \mathcal{U}_i \cap \mathcal{U}_j \to PU(\mathcal{H}) = Aut(\mathcal{K})$. Consistency of (3) on triple overlaps $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$ implies that

$$g_{ij} g_{jk} g_{ki} = \zeta_{ijk}, \tag{4}$$

where ζ_{ijk} are $U(1)$-valued functions. One verifies that on quadruple overlaps $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_l$, the functions ζ_{ijk} satisfy

$$\zeta_{ijk} \zeta_{ikl} = \zeta_{jkl} \zeta_{ijl}. \tag{5}$$

Therefore we see that on quadruple overlaps $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_l$, one has

$$\log \zeta_{ijk} + \log \zeta_{ikl} - \log \zeta_{jkl} - \log \zeta_{ijl} = 2\pi\sqrt{-1}\kappa_{ijkl} \tag{6}$$

where $\{\kappa_{ijkl}\}$ is a $\mathbb{Z}$-valued Čech 3-cocycle, and therefore defines an element $\kappa \in H^3(X, \mathbb{Z})$. This is the Dixmier-Douady class $\delta(\mathcal{E})$ mentioned in the paragraph above.

Let $Tors(H^3(X, \mathbb{Z}))$ denote the subgroup of torsion elements in $H^3(X, \mathbb{Z})$. Suppose now that X is compact. Then there is a well known description of $Tors(H^3(X, \mathbb{Z}))$ in terms of finite dimensional Azumaya algebras over X [DK]. Recall that an Azumaya algebra of rank m over X is a locally trivial algebra bundle over X whose fibre is isomorphic to the algebra of $m \times m$ matrices $M_m(\mathbb{C})$. An example of an Azumaya algebra over X is the algebra $\mathrm{End}(E)$ of all endomorphisms of a vector bundle E over X. Two Azumaya algebras $\mathcal{E}$ and $\mathcal{F}$ over X are said to be equivalent if there are

vector bundles E and F over X such that $\mathcal{E}\otimes\mathrm{End}(E)$ is isomorphic to $\mathcal{F}\otimes\mathrm{End}(F)$. In particular, an Azumaya algebra of the form $\mathrm{End}(E)$ is equivalent to $C(X)$ for any vector bundle E over X. The group of all equivalence classes of Azumaya algebras over X is called the Brauer group of X and is denoted by $Br(X)$. We will denote by $\delta'(\mathcal{E})$ the class in $Tors(H^3(X,\mathbb{Z}))$ corresponding to the Azumaya algebra $\mathcal{E}$ over X. It is constructed by analogy to the local coordinate description given in the previous paragraph. Serre's theorem asserts that $Br(X)$ and $Tors(H^3(X,\mathbb{Z}))$ are isomorphic.

Thus we see that there are two distinct descriptions of $Tors(H^3(X,\mathbb{Z}))$, one in terms of finite dimensional Azumaya algebras over X, and the other is terms of locally trivial bundles over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$. These two descriptions are related as follows. Given an Azumaya algebra $\mathcal{E}$ over X, then the tensor product $\mathcal{E}\otimes\mathcal{K}$ is a locally trivial bundle over X with fibre $M_m(\mathbb{C})\otimes\ \mathcal{K}\cong\mathcal{K}$ and structure group $Aut(\mathcal{K})$, such that $\delta'(\mathcal{E})=\delta(\mathcal{E}\otimes\mathcal{K})$. Notice that the algebras $C(X,\mathcal{E})$ and $C(X,\mathcal{E}\otimes\mathcal{K})=C(X,\mathcal{E})\otimes\mathcal{K}$ are Morita equivalent. Moreover if $\mathcal{E}$ and $\mathcal{F}$ are equivalent Azumaya algebras over X, then $\mathcal{E}\otimes\mathcal{K}$ and $\mathcal{F}\otimes\mathcal{K}$ are isomorphic locally trivial bundles over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$. To see this, we recall that $\mathcal{K}=\lim_n M_n(\mathbb{C})$ where the limit is taken in the C^*-norm topology [Dix]. Since the automorphism group of $M_n(\mathbb{C})$ is $PU(n)=SU(n)/\mathbb{Z}_n$ and $Aut(\mathcal{K})=PU(\mathcal{H})$, it is in this sense that $\lim_{n\to\infty}SU(n)/\mathbb{Z}_n=PU(\mathcal{H})$. The equivalence relation for the Azumaya algebras $\mathcal{E}$ and $\mathcal{F}$ becomes, $\mathcal{E}\otimes\mathcal{K}(E)$ and $\mathcal{F}\otimes\mathcal{K}(F)$ are isomorphic, where $\mathcal{K}(E)$ and $\mathcal{K}(F)$ are the bundles of compact operators on the infinite dimensional Hilbert bundles E and F, respectively. By Kuiper's theorem, the group $U(\mathcal{H})$ of unitary operators in an infinite dimensional Hilbert space $\mathcal{H}$ is contractible in the strong operator topology. Therefore, the infinite dimensional Hilbert bundles E and F are trivial, and therefore both $\mathcal{K}(E)$ and $\mathcal{K}(F)$ are isomorphic to $X\times\mathcal{K}$. It follows that $\mathcal{E}\otimes\mathcal{K}(E)$ and $\mathcal{F}\otimes\mathcal{K}(F)$ are isomorphic if and only if $\mathcal{E}\otimes\mathcal{K}$ and $\mathcal{F}\otimes\mathcal{K}$ are isomorphic, as asserted.

Now, suppose that $\mathcal{E}$ is a locally trivial bundle over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$, such that $\delta(\mathcal{E})$ is a torsion element, then there is a positive integer n such that $0=n\delta(\mathcal{E})=\delta(\mathcal{E}^{\otimes n})$. Therefore $\mathcal{E}^{\otimes n}$ is isomorphic to the trivial bundle $X\times\mathcal{K}$, and it follows that $\mathcal{E}$ has transition functions $g_{ij}:\mathcal{U}_i\cap\mathcal{U}_j\to Aut(\mathcal{K})$ that are locally constant functions. That is, $\mathcal{E}$ is a *flat* locally trivial bundle over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$, which is given by a representation of $\pi_1(X)$ into $Aut(\mathcal{K})$. Therefore we see that $Tors(H^3(X,\mathbb{Z}))$ parametrizes the topologically nontrivial isomorphism classes of *flat* locally trivial bundles over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$. In fact, given a representation $\rho:\pi_1(X)\to Aut(\mathcal{K})=PU(\mathcal{H})$, there is a map $\lambda:\pi_1(X)\to U(\mathcal{H})$, such that $\rho(\gamma)=Ad(\lambda(\gamma))$ for all $\gamma\in\pi_1(X)$, satisfying the identity

$$\lambda(\gamma_1)\lambda(\gamma_2)\lambda(\gamma_1\gamma_2)^{-1}=\sigma(\gamma_1,\gamma_2),\qquad\forall\gamma_1,\gamma_2\in\pi_1(X),\tag{7}$$

where $\sigma : \pi_1(X) \times \pi_1(X) \to U(1)$ satisfies the cocycle identity

$$\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_2, \gamma_3)\sigma(\gamma_1, \gamma_2\gamma_3), \qquad \forall \gamma_1, \gamma_2, \gamma_3 \in \pi_1(X) \tag{8}$$

and is normalized, $1 = \sigma(1, \gamma) = \sigma(\gamma, 1)$, $\forall \gamma \in \pi_1(X)$. Such a normalized $U(1)$-valued group 2-cocycle on $\pi_1(X)$ is called a *multiplier* on $\pi_1(X)$. The flat bundle defined by ρ is $\mathcal{E}_\rho = \left(\widetilde{X} \times \mathcal{K}\right) / \sim$, where $(x, T) \sim (\gamma^{-1}x, \rho(\gamma)T)$ for all $\gamma \in \pi_1(X)$ and $\widetilde{X}$ is the universal cover of X. Then $\mathcal{E}_\rho$ has Dixmier-Douady invariant $\delta(\mathcal{E}_\rho) = \delta''(f^*\sigma)$ [Was], where δ'' is the connecting homomorphism in the "change of coefficients" long exact sequence

$$\cdots \to H^2(\pi_1(X), \mathbb{R}) \to H^2(\pi_1(X), U(1)) \xrightarrow{\delta''} H^3(\pi_1(X), \mathbb{Z}) \to \cdots$$

that is associated to the short exact sequence of coefficient groups,

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$$

and $f : X \to B\pi_1(X)$ denotes the continuous map classifying the universal cover.

We will now show that a closed, integral, $\pi_1(X)$-invariant differential 2-form ω on the universal cover $\widetilde{X}$ of X determines such a multiplier on $\pi_1(X)$. [In the quantum Hall effect ω represents the magnetic field, cf. [CHMM, CHM].]

By geometric pre-quantization, there is an essentially unique Hermitian line bundle $\mathcal{L} \to \widetilde{X}$ and a Hermitian connection ∇ whose curvature is $\nabla^2 = i\omega$ ($i = \sqrt{-1}$). Since ω is $\pi_1(X)$-invariant, one sees that for $\gamma \in \pi_1(X)$, $\gamma^*\nabla$ is also a Hermitian vector potential for ω, i.e., $(\gamma^*\nabla)^2 = i\gamma^*\omega = i\omega$. Now $\gamma^*\nabla - \nabla = iA_\gamma \in \Omega^1(\widetilde{X}, \mathbb{R})$. Since $i\omega = \nabla^2 = (\gamma^*\nabla)^2$, we see that $0 = \nabla A_\gamma + A_\gamma\nabla = dA_\gamma$, i.e. A_γ is a closed 1-form on the simply-connected manifold $\widetilde{X}$. Therefore it is exact, i.e. $A_\gamma = d\phi_\gamma$ $(*)$, where φ_γ is a real-valued smooth function on $\widetilde{X}$. It is easy to see that it also satisfies

(i) $\varphi_\gamma(x) + \varphi_{\gamma'}(\gamma^{-1}x) - \varphi_{\gamma'\gamma}(x)$ is independent of $x \in \widetilde{X}$ for all $\gamma, \gamma' \in \pi_1(X)$;
(ii) $\varphi_\gamma(x_0) = 0$ for some $x_0 \in \widetilde{X}$ and for all $\gamma \in \pi_1(X)$.

Equation (i) follows immediately from $(*)$ and (ii) is a normalization. For $\gamma \in \pi_1(X)$ define $U_\gamma f(x) = f(\gamma^{-1} \cdot x)$ (translations) and $S_\gamma f(x) = e^{i\varphi_\gamma(x)} f(x)$ (phase) and $T_\gamma = U_\gamma \circ S_\gamma$ (magnetic translations).

Then one computes that

$$T_{\gamma_1} T_{\gamma_2} = \sigma(\gamma_1, \gamma_2) T_{\gamma_1\gamma_2}\,, \qquad \text{for } \gamma_1, \gamma_2 \in \pi_1(X)\,, \tag{9}$$

where $\sigma : \pi_1(X) \times \pi_1(X) \to U(1)$ is defined as $\sigma(\gamma_1, \gamma_2) = e^{-i\phi_{\gamma_1}(\gamma_2^{-1} \cdot x_0)}$, $\forall \gamma_1, \gamma_2 \in \pi_1(X)$. It satisfies the cocycle condition (8) by the associativity of T. Thus σ is the multiplier on $\pi_1(X)$ that is determined by ω and a choice of base-point x_0. Any other choice of base-point determines a cohomologous multiplier on $\pi_1(X)$.

Conversely, given a multiplier σ on $\pi_1(X)$, consider the Hilbert space of square summable functions on $\pi_1(X)$,

$$\ell^2(\pi_1(X)) = \left\{ f : \pi_1(X) \to \mathbb{C} : \sum_{\gamma \in \pi_1(X)} |f(\gamma)|^2 < \infty \right\} . \tag{10}$$

The left σ-regular representation on $\ell^2(\pi_1(X))$ is defined as being, $\quad \forall\, \gamma, \gamma' \in \pi_1(X)$,

$$\begin{aligned} L^\sigma \quad &: \pi_1(X) \longrightarrow U(\ell^2(\pi_1(X))) \\ (L^\sigma_\gamma f)(\gamma') \quad &= f(\gamma^{-1}\gamma')\sigma(\gamma, \gamma^{-1}\gamma'). \end{aligned} \tag{11}$$

It satisfies $L^\sigma_\gamma L^\sigma_{\gamma'} = \sigma(\gamma,\gamma')L^\sigma_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \pi_1(X)$. That is, the left σ-regular representation L^σ on $\ell^2(\pi_1(X))$ is a projective unitary representation. Therefore $Ad(L^\sigma) : \pi_1(X) \longrightarrow PU(\ell^2(\pi_1(X))) = Aut(\mathcal{K})$ is a representation of $\pi_1(X)$ into $Aut(\mathcal{K})$, and so determines a flat locally trivial bundle $\mathcal{E}_\sigma$ over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$. It follows by a result in [Was] that the Dixmier-Douady invariant of $\mathcal{E}_\sigma$ is $\delta(\mathcal{E}_\sigma) = \delta''(f^*\sigma)$, where f and δ'' are the same as above.

3. Twisted K-theory and noncommutative geometry

Let X be a locally compact, Hausdorff space with a countable basis of open sets, for example a smooth manifold. Let $[H] \in H^3(X,\mathbb{Z})$. Then the *twisted K-theory* was defined by Rosenberg [Ros] as

$$K^j(X,[H]) = K_j(C_0(X,\mathcal{E}_{[H]})) \qquad j = 0, 1, \tag{12}$$

where $\mathcal{E}_{[H]}$ is the unique locally trivial bundle over X with fibre $\mathcal{K}$ and structure group $Aut(\mathcal{K})$ such that $\delta(\mathcal{E}_{[H]}) = [H]$, and $K_\bullet(C_0(X,\mathcal{E}_{[H]}))$ denotes the topological K-theory of the C^*-algebra of continuous sections of $\mathcal{E}_{[H]}$ that vanish at infinity. Notice that when $[H] = 0 \in H^3(X,\mathbb{Z})$, then $\mathcal{E}_{[H]} = X \times \mathcal{K}$, therefore $C_0(X,\mathcal{E}_{[H]}) = C_0(X) \otimes \mathcal{K}$ and by Morita invariance of K-theory, the twisted K-theory of X coincides with the standard K-theory of X in this case. The Morita invariance of the K-theory of a C^*-algebra $\mathcal{A}$ can be explained as follows. $K_0(\mathcal{A})$ can be defined as the Grothendieck group of Murray-von Neumann equivalence classes of projections in $\mathcal{A} \otimes \mathcal{K}$ (cf. [WO]). Therefore the isomorphism $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$ induces the isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{A} \otimes \mathcal{K})$. Also $K_1(\mathcal{A})$ can be defined as the path components of the group $\{g \in U((\mathcal{A} \otimes \mathcal{K})^+) : g - 1 \in \mathcal{A} \otimes \mathcal{K}\}$ in the norm topology, where $(\mathcal{A} \otimes \mathcal{K})^+$ denotes the C^* algebra obtained from $\mathcal{A} \otimes \mathcal{K}$ by adjoining the identity operator. Therefore we again see that the isomorphism $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$ induces the isomorphism $K_1(\mathcal{A}) \cong K_1(\mathcal{A} \otimes \mathcal{K})$.

When X is compact and when $[H] \in Tors(H^3(X,\mathbb{Z}))$, then there is an alternate description of the twisted K-theory $K^j(X,[H])$, due to Donovan and Karoubi [DK], as being the topological K-theory of the algebra of sections of an Azumaya algebra

$\mathcal{F}$ over X with $\delta'(\mathcal{F}) = [H]$. Notice that this is well defined as the C^*-algebra of sections over any other equivalent Azumaya algebra is actually Morita equivalent to $C(X, \mathcal{F})$, and therefore they have the same K-theory. The relation between the Donovan-Karoubi twisted K-theory and the Rosenberg twisted K-theory is obtained by tensoring the Azumaya algebra with $\mathcal{K}$, as discussed in Section 2, and by Morita invariance of K-theory we see that the two definitions for the twisted K-theory of X are isomorphic .

There are alternate descriptions of $K^0(X, [H])$, whose elements are generated by projections in $C_0(X, \mathcal{E}_{[H]}) \otimes \mathcal{K} \cong C_0(X, \mathcal{E}_{[H]})$. It is proved in [Ros] that when X is compact one has

$$K^0(X, [H]) = [Y, U(\mathcal{Q})]^{Aut(\mathcal{K})} \tag{13}$$

where Y is the principal $Aut(\mathcal{K})$ bundle over X such that $\mathcal{E} = Y \times_{Aut(\mathcal{K})} \mathcal{K}$ and $U(\mathcal{Q})$ denotes the group of unitary elements in the Calkin algebra $\mathcal{Q} = B(\mathcal{H})/\mathcal{K}$ in the norm topology, where $B(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. Another result in [Ros] is that when X is compact one has

$$K^1(X, [H]) \ = \ [Y, U(\mathcal{H})]^{Aut(\mathcal{K})} \, . \tag{14}$$

We will next give a "local coordinate" description of objects in the twisted K-theory.

Since projections in $\mathcal{K}$ have finite dimensional ranges [Dix], we observe that even though the algebra $C_0(X, \mathcal{E}_{[H]})$ is infinite dimensional, the projections in $C_0(X, \mathcal{E}_{[H]})$ have ranges which are bundle like objects with finite dimensional fibres. These can be described as follows. Assume first that X is compact. Recall that projection P in the the algebra $C_0(X, \mathcal{E}_{[H]})$ satisfies $P^* = P = P^2$. That is, P is a is a collection of continuous functions $P_i : \mathcal{U}_i \to \mathcal{K}$ such that $P_i^* = P_i = P_i^2$ and such that on the overlaps $\mathcal{U}_i \cap \mathcal{U}_j$, one has

$$P_i = Ad(g_{ij}) P_j. \tag{15}$$

Here the continuous functions $Ad(g_{ij}) : \mathcal{U}_i \cap \mathcal{U}_j \to PU(\mathcal{H}) = Aut(\mathcal{K})$ are the transition functions for the locally trivial bundle $\mathcal{E}_{[H]}$ with fibre $\mathcal{K}$, where $g_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \to U(\mathcal{H})$ are continuous functions on the overlaps, satisfying $g_{ij} g_{ji} = 1$, and equation (4). Observe that $(Ad(g_{ij})P_j)^* = Ad(g_{ij})P_j = (Ad(g_{ij})P_j)^2$. We see that the range of the projection $P_i(x)$ is a *finite dimensional* subspace $V_{i,x} \subset \mathcal{H}$ for each $x \in \mathcal{U}_i$, and the collection $\{V_{i,x}\}_{x \in \mathcal{U}_i}$ is continuous over $\mathcal{U}_i$ in the sense that $\mathcal{U}_i \ni x \to P_i(x)$ is continuous. On the overlaps $\mathcal{U}_i \cap \mathcal{U}_j$, an element $v \in V_{i,x}$ is identified with the element $g_{ji}(x) v \in V_{j,x}$. We will then say that the data $\{\mathcal{U}_i, \{V_{i,x}\}_{x \in \mathcal{U}_i}, g_{ij}\}$ defines a *gauge-bundle* over X, to be consistent with terminology in the physics literature. The definition is incomplete in the sense that it does not discuss the dependence on the choices made. However this can be remedied by introducing sheaves of categories [Br] and will be done elsewhere. Note that a gauge-bundle is not a manifold in

general, unlike the definition of a vector bundle. Therefore, we see that a projection P in $C_0(X, \mathcal{E}_{[H]})$ defines a gauge-bundle over X. Recall that two projections P and Q in $C_0(X, \mathcal{E}_{[H]})$ are Murray-von Neumann equivalent if there is a $\Lambda \in C_0(X, \mathcal{E}_{[H]})$ such that $P = \Lambda^*\Lambda$ and $Q = \Lambda\Lambda^*$. In local coordinates, this means that there is a collection of continuous functions $\Lambda_i : \mathcal{U}_i \to \mathcal{K}$ such that $P_i = \Lambda_i^*\Lambda_i$ and $Q_i : \Lambda_i\Lambda_i^*$, such that on the overlaps $\mathcal{U}_i \cap \mathcal{U}_j$, one has

$$\Lambda_i = Ad(g_{ij})\Lambda_j. \tag{16}$$

In terms of gauge-bundles, this means that if $\{\mathcal{U}_i, \{V_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}$ and $\{\mathcal{U}_i, \{W_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}$ are the gauge-bundles defined by the projections P and Q respectively, then an *isomorphism* of gauge bundles is given by such an element Λ. Note that $\Lambda_i(x) : V_{i,x} \to W_{i,x}$ is an isomorphism of finite dimensional vector spaces, with inverse $\Lambda_i^*(x) : W_{i,x} \to V_{i,x}$. The direct sum of gauge-bundles $\{\mathcal{U}_i, \{V_{i,x}\}_{x\in U_i}, g_{ij}\}$ and $\{\mathcal{U}_i, \{W_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}$ is again a gauge-bundle

$$\{\mathcal{U}_i, \{V_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\} \oplus \{\mathcal{U}_i, \{W_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\} = \{\mathcal{U}_i, \{V_{i,x} \oplus W_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}\,. \tag{17}$$

It corresponds to taking the orthogonal direct sum of the projections that define the gauge-bundles. Note that the gauge-bundles $\{\mathcal{U}_i, \{V_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\} \oplus \{\mathcal{U}_i, \{W_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}$ and $\{\mathcal{U}_i, \{W_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\} \oplus \{\mathcal{U}_i, \{V_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}$ are naturally isomorphic. Define $\mathbf{Vect}(X, [H])$ to be the Abelian semigroup of isomorphism classes of gauge-bundles $[\{\mathcal{U}_i, \{V_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}]$ with the direct sum operation. Then the associated Grothendieck group is just $K^0(X, [H])$. That is, if X is compact,

$$\begin{aligned} K^0(X,[H]) \;\; = \;\; & \Big\{[\{\mathcal{U}_i, \{V_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}] - [\{\mathcal{U}_i, \{W_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}] : \{\mathcal{U}_i, \{V_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}, \\ & \{\mathcal{U}_i, \{W_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\} \text{ are gauge–bundles over } X\Big\} \end{aligned}$$

When X is not compact, then $K^0(X, [H])$ consists of isomorphism classes of triples $(\{\mathcal{U}_i, \{V_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}, \{\mathcal{U}_i, \{W_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}, \Lambda)$, where $\{\mathcal{U}_i, \{V_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}$ and $\{\mathcal{U}_i, \{W_{i,x}\}_{x\in\mathcal{U}_i}, g_{ij}\}$ are gauge-bundles over X and Λ is an isomorphism between the gauge bundles on the complement of a compact subset of X. There is a more elegant description of such objects using an analogue of Quillen's formalism [Qu]. This, and an analogous description for $K^1(X, [H])$, will be given elsewhere.

4. Conclusions

In this paper we have proposed a natural candidate for the classification of D-brane charges, in the presence of a topologically non-trivial B-field, in terms of the twisted K-theory of certain infinite-dimensional, locally trivial, algebra bundles of compact operators. We have also shown that in the case of torsion elements $[H]$ our proposal is equivalent to that of Witten [Wi]. While the necessity of incorpating nontorsion classes $[H] \in H^3(X, \mathbb{Z})$ in the formalism has forced us to consider infinite dimensional algebra bundles, this description is, in some respects, more natural even

for torsion elements. The reasons are, first of all, that the bundle $\mathcal{E}_{[H]}$ is the *unique* locally trivial bundle over X such that $\delta(\mathcal{E}_{[H]}) = [H]$, while the corresponding Azumaya algebras are only determined upto equivalence. And, secondly, our proposal holds for any, locally compact, Hausdorff space with a countable basis of open sets, while the twisted K-theory of an Azumaya algebra is only defined in the case of compact X.

It is clear that our proposal needs further study [BM]. For instance, by using a fundamental theorem of Grothendieck [Gr], it can be shown that in the case of torsion elements $[H]$, the groups $K^\bullet(X)$ and $K^\bullet(X,[H])$ are rationally equivalent [Wi]. The proof appears to break down for nontorsion elements, however. The structure of $K(X,[H])$, as well as its physical interpretation, would be greatly elucidated by studying the Connes-Chern map $K_\bullet(\mathcal{A}) \to HPC_\bullet(\mathcal{A})$ (see [Co]) and a possible relation of the periodic cyclic homology $HPC_\bullet(\mathcal{A})$ to the de-Rham cohomology for the noncommutative algebra $\mathcal{A} = C_0^\infty(X, \mathcal{E}_{[H]})$ of smooth sections vanishing to all orders at infinity. Note that by the analogue of Oka's principle in this context, one has $K_\bullet(\mathcal{A}) \cong K^\bullet(X,[H])$.

Other issues, such as the cancellation of global string worldsheet anomalies, discrete torsion in this setting and examples of D-branes in NS-charged backgrounds remain to be worked out as well.

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References

[BM] P. Bouwknegt and V. Mathai, work in progress.

[Br] J.-L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Progress in Mathematics **107**, (Birkhäuser, Boston, 1993).

[CHMM] A. Carey, K. Hannabuss, V. Mathai and P. McCann, *Quantum Hall Effect on the hyperbolic plane*, Commun. Math. Physics, **190** no. 3 (1997) 629–673, [`dg-ga/9704006`].

[CHM] A. Carey, K. Hannabuss, V. Mathai, *Quantum Hall effect on the hyperbolic plane in the presence of disorder*, Lett. Math. Phys. **47** (1999), no. 3, 215–236, [`math.DG/9804128`].

[Co] A. Connes, *Noncommutative geometry*, (Academic Press, San Diego, 1994).

[CDS] A. Connes, M. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: Compactification on tori*, JHEP **02** (1998) 003, [`hep-th/9711162`].

[Dix] J. Dixmier, *C^*-algebras and their representations*, (North Holland, Amsterdam, 1982).

[DD] J. Dixmier and A. Douady, *Champs continues d'espaces hilbertiens at de C^*-algèbres*, Bull. Soc. Math. France **91** (1963) 227–284.

[DK] P. Donovan and M. Karoubi, *Graded Brauer groups and K-theory with local coefficients*, Inst. Hautes Études Sci. Publ. Math., **38** (1970) 5–25.

[FW] D. Freed and E. Witten, *Anomalies in string theory with D-branes*, [hep-th/9907189].

[Gr] A. Grothendieck, *Le groupe de Brauer, I, II, III. 1968 Dix Exposés sur la Cohomologie des Schémas*, pp. 46–188, (North-Holland, Amsterdam; Masson, Paris).

[Ka] A. Kapustin, *D-branes in a topologically nontrivial B-field*, [hep-th/9909089].

[MM] R. Minasian and G. Moore, *K-theory and Ramond-Ramond charge*, JHEP **11** (1997) 002, [hep-th/9710230].

[MW] G. Moore and E. Witten, *Self-duality, Ramond-Ramond fields, and K-theory*, [hep-th/9912279].

[Pa] E.M. Parker, *The Brauer group of graded continuous trace C^*-algebras*, Trans. Amer. Math. Soc., **308** (1988), no. 1, 115–132.

[Qu] D. Quillen, *Superconnections and the Chern character*, Topology **24** (1985), no. 1, 89-95.

[RW] I. Raeburn and D. Williams, *Morita equivalence and continuous trace C^*-algebras*, Math. Surv. and Mono., **60** (1998).

[Ros] J. Rosenberg, *Continuous trace algebras from the bundle theoretic point of view*, Jour. Austr. Math. Soc., **47** (1989), 368–381.

[SW] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **09** (1999) 032, [hep-th/9908142].

[Se] A. Sen, *Stable non-BPS states in string theory*, JEHP **06** (1998) 007, [hep-th/9803194]; *ibid.*, *Stable non-BPS bound states of BPS D-branes*, JHEP **08** (1998) 010, [hep-th/9805019]; *ibid.*, *Tachyon condensation on the brane antibrane system*, JHEP **08** (1998) 012, [hep-th/9805170]; *ibid.*, *$SO(32)$ spinors of type I and other solitons on brane-antibrane pair*, JHEP **09** (1998) 023, [hep-th/9808141].

[Was] A. Wassermann, *Automorphic actions of compact groups on operator algebras*, Ph.D. dissertation, Univ. of Pennsylvannia, Philadelphia, 1981; *ibid.*, *Equivariant K-theory I: compact transformation groups with one orbit type*, IHES preprint, 1985.

[WO] N.E. Wegge-Olsen, *K-theory and C^*-algebras: A friendly approach*, (Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993).

[Wh] G. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics **61**, (Springer-Verlag, New York-Berlin, 1978).

[Wi] E. Witten, *D-branes and K-theory*, JHEP **12** (1998) 019, [hep-th/9810188].

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